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Laws Governing Isolated Horizons: Inclusion of Dilaton Couplings

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Abstract

Mechanics of non-rotating black holes was recently generalized by replacing the static event horizons used in standard treatments with 'isolated horizons.' This framework is extended to incorporate dilaton couplings. Since there can be gravitational and matter radiation outside isolated horizons, now the fundamental parameters of the horizon, used in mechanics, must be defined using only the *local* structure of the horizon, without reference to infinity. This task is accomplished and the zeroth and first laws are established. To complement the previous work, the entire discussion is formulated tensorially, without any reference to spinors.

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I. INTRODUCTION

The zeroth and first laws of black hole mechanics refer to equilibrium situations and small departures therefrom. Therefore, in the standard treatments [1–5] one restricts oneself to stationary space-times admitting event horizons and perturbations off such space-times. While this simple idealization is a natural starting point, from physical considerations it seems overly restrictive. (See [6,7] and especially [8] for a detailed discussion.). A framework, which is tailored to more realistic physical situations was introduced in [6] and the zeroth and first laws were correspondingly extended in [7,8]. This framework generalizes black hole mechanics in two directions. First, the notion of event horizons is replaced by that of 'isolated horizons'; while the former can only be defined retroactively, after having access to the entire space-time history, the latter can be defined quasi-locally. Second, the underlying space-time need not admit any Killing field; isolated horizons need not be Killing horizons. The static event horizons normally used in black hole mechanics [1–3] and the cosmological horizons in de Sitter space-times [9] are all special cases of isolated horizons. However, because we can now admit gravitational and matter radiation, there are many more examples. In particular, while the space $\mathcal{S}$ of static space-times admitting event horizons in the Einstein-Maxwell theory is finite dimensional, the space $\mathcal{IH}$ of space-times admitting isolated horizons is *infinite* dimensional [8].

In the discussion of zeroth and first laws in [8], two restrictive assumptions were made. First, the horizon was assumed to be non-rotating. Second, while a rather general class of matter fields was allowed, it was assumed that the only relevant charges —i.e., hair— are the standard electric and magnetic ones. In this note, we will continue to work with non-rotating horizons but weaken the second assumption by allowing a dilaton field and a general coupling, parameterized as usual by a (non-negative) coupling constant α. By setting $\alpha = 0$, one recovers the standard Einstein-Maxwell-Klein-Gordon system (which was included in [8]). For $\alpha = 1$ (in units in which the string tension is set equal to one) one obtains the action widely used in the study of the low energy limit of string theory. For all values of α, we will be able to extend the isolated horizon framework and prove the zeroth and first laws.

The main challenge encountered in this extension is the following. In the standard treatment, static black holes in dilaton gravity are characterized by three parameters: the mass M, the (modified) electric charge $\tilde{Q}$ (or, the magnetic charge P) and a dilaton charge ϕ_∞. Of these, M and ϕ_∞ are defined at *infinity* (while $\tilde{Q}$ and P, being absolutely conserved, can be defined on any 2-sphere). In the more general context of isolated horizons, gravitational radiation and matter fields may be present between the horizon and infinity. Now M and ϕ_∞ carry information about these fields and can no longer be regarded as intrinsic parameters of the horizon itself. Therefore, the laws of mechanics can not refer to them; we need new intrinsic parameters. In the Einstein-Maxwell case, this was accomplished [7,8] by replacing M with the area a_Δ of the isolated horizon Δ (in this case, $\phi_\infty = 0$). A natural strategy now is to continue to use a_Δ in place of M and use the value ϕ_Δ of the dilaton field *on* the horizon in place of ϕ_∞. However, it is far from obvious that this strategy is viable. For, there are at least four conditions to meet. First, the isolated horizon boundary conditions should tell us that the field ϕ_Δ is constant on Δ. Second, we should be able to define surface gravity κ in absence of a Killing field and show that it is constant on *any* isolated horizon, in spite of the

fact that the values of fields such as the Weyl component Ψ_4 and the Maxwell component ϕ_2 can be 'dynamical' on isolated horizons. Third, to have a meaningful formulation of the first law, the Hamiltonian strategy of [8] for defining the mass M_Δ of the isolated horizon should go through and enable us to express M_Δ in terms only of the horizon parameters, in spite of the fact that values of fields such as Ψ_4 and Φ_2 on Δ are not determined by the horizon parameters. Finally, the functional dependence of the horizon mass M_Δ, the surface gravity κ and the electric potential Φ on the horizon parameters $a_\Delta, \tilde{Q}_\Delta$ (or, P_Δ) and ϕ_Δ should be such that the first law holds. We will show that all these conditions can be met.

The organization of the paper is as follows. For convenience of readers who may not be familiar with dilaton couplings, in Section II we briefly recall the relevant results from literature. In Section III, we specify the boundary conditions defining non-rotating isolated horizons in dilaton gravity and summarize their consequences which are needed for the main results. Section IV introduces surface gravity κ and establishes the zeroth law while Section V introduces the mass M_Δ and establishes the first law. Since the underlying strategy in sections IVand V is the same as in [8], we will use the same notation (which, incidentally, differs in some minor ways from the conventions used in [6]) and refrain from repeating the detailed arguments and proofs given in [8]. Rather, we will recall the main conceptual steps and emphasize the new issues and subtleties that arise due to dilaton couplings. Also, we will use this opportunity to make some new observations which hold also in cases considered in [8]. Finally, while the earlier treatments were spinorial, in this paper, we will complement that discussion by working only with tensors.

II. DILATON COUPLING AND STATIC BLACK HOLES

A. Einstein-Maxwell-Dilaton System

In dilaton gravity, (in the so-called Einstein frame) the gravitational part of the action, $S_{\rm Grav}$ is the standard one [6,8]. The matter part of the action may be less familiar. It is given by:

$$S_{\rm Dil}(\phi, A) = -\frac{1}{16\pi}\int_{\mathcal{M}} \sqrt{-g}[2(\nabla\phi)^2 + e^{-2\alpha\phi}\mathbf{F}_{ab}\mathbf{F}^{ab}]\mathrm{d}^4x \tag{2.1}$$

where α is a free parameter which governs the strength of the coupling of the dilaton field ϕ to the Maxwell field $\mathbf{F}_{ab}$. Since the action is invariant under simultaneous changes of sign of ϕ and α, without loss of generality, one can restrict oneself to non-negative values of α. If $\alpha = 0$, we recover the Einstein-Maxwell-Klein-Gordon system. The total action $S_{\rm Tot}$ is given by

$$S_{\rm Tot} = S_{\rm Grav} + S_{\rm Dil}. \tag{2.2}$$

The equations of motion that follow from $S_{\rm Tot}$ are:

$$\nabla_a {}^\star\mathbf{F}^{ab} = 0, \qquad \nabla_a(e^{-2\alpha\phi}\mathbf{F}^{ab}) = 0, \tag{2.3}$$

$$\nabla^2\phi + \frac{\alpha}{2}e^{-2\alpha\phi}\mathbf{F}^2 = 0, \quad \text{and} \tag{2.4}$$

$$R_{ab} = G\left[2\nabla_a\phi\nabla_b\phi + 2e^{-2\alpha\phi}\mathbf{F}_{ac}\mathbf{F}_b{}^c - \frac{1}{2}g_{ab}e^{-2\alpha\phi}\mathbf{F}^2\right], \tag{2.5}$$

where $\mathbf{F}^2 = \mathbf{F}_{ab}\mathbf{F}^{ab}$.

Since $\mathrm{d}\mathbf{F} = 0$ by definition of $\mathbf{F}$, the magnetic charge,

$$P[S] := \frac{1}{4\pi} \oint_S \mathbf{F} \tag{2.6}$$

is absolutely conserved, i.e., is independent of the particular choice of the 2-sphere S, in a given homology class, made in its evaluation. Note however that $\mathrm{d}\,{}^\star\mathbf{F} \neq 0$, whence the standard electric charge

$$Q_{\mathrm{std}} := \frac{1}{4\pi} \oint_S {}^\star\mathbf{F} \tag{2.7}$$

(with ${}^\star\mathbf{F}_{ab} = \frac{1}{2}\,\epsilon_{abcd}\mathbf{F}^{cd}$) is *not* absolutely conserved. The effective current J^a depends on the Maxwell field itself and is given by $J^a = 2\alpha\,[\exp -2\alpha\phi]\,\nabla_b\phi\,F^{ab}$. Hence, Q_{std} satisfies a balance equation:

$$Q_{\mathrm{std}}[S_1] - Q_{\mathrm{std}}[S_2] = \frac{\alpha}{2\pi} \int_{M_{12}} \mathrm{d}\phi \wedge {}^\star\mathbf{F} \tag{2.8}$$

where M_{12} is the 3-surface with boundaries S_1 and S_2. However, it is obvious from (2.3) that the 2-form $(\exp\, -2\alpha\phi)\ {}^\star\mathbf{F}$ is closed. Therefore, there is in fact a conserved electric charge,

$$Q_{\mathrm{cons}}[S] := \frac{1}{4\pi} \oint_S e^{-2\alpha\phi}\ \ {}^\star\mathbf{F}\,. \tag{2.9}$$

Again, $Q_{\mathrm{cons}}[S]$ depends only on the homology class of S. Finally, in the asymptotically flat context of primary interest to this paper, the requirement that the 4-momentum be finite leads one to a natural boundary condition on the dilaton field at spatial infinity: ϕ must approach a constant, ϕ_∞. This constant provides a 'dilatonic charge' ϕ_∞ associated with the solution. [1] Note that the values of $Q_{\mathrm{std}}[S_\infty]$,$Q_{\mathrm{cons}}[S_\infty]$ and ϕ_∞ are related via

$$Q_{\mathrm{cons}}[S_\infty] = e^{-2\alpha\phi_\infty}\ Q_{\mathrm{std}}[S_\infty] \tag{2.10}$$

so that, in the parameterization of the solution, one can replace ϕ_∞ with $Q_{\mathrm{std}}[S_\infty]$.

Of special interest are two symmetries of the theory. The first is a natural generalization of the standard duality rotation of the Maxwell theory:

$$(\phi, \mathbf{F}_{ab}, g_{ab}) \ \mapsto\ \mathcal{D}\,(\phi, \mathbf{F}_{ab}, g_{ab}) = (-\phi, {}^\star\mathbf{F}_{ab}, g_{ab})\,. \tag{2.11}$$

As in the standard Maxwell theory, the action fails to be invariant under this transformation since $(\exp\, -2\alpha\phi)\ \mathbf{F}^2 \ \mapsto\ -(\exp -2\alpha\phi)\ \mathbf{F}^2$. However, it is obvious from the equations of motion that if $(\phi, \mathbf{F}, g)$ is a solution, so is the duality rotated triplet $\mathcal{D}\,(\phi, \mathbf{F}, g)$. This fact is generally exploited to restrict oneself to the sector of the theory in which the magnetic

[1] In the literature, dilatonic charge D is generally defined only in stationary space-times. The new information in D, not already contained in the mass M and charge Q_{cons}, is ϕ_∞.

charge vanishes: Properties of solutions with a magnetic charge can be obtained simply by a duality rotation of the solution with a corresponding electric charge.[2]

The second symmetry is continuous and corresponds to a constant shift in ϕ:

$$(\phi, \mathbf{F}_{ab}, g_{ab}) \mapsto \mathcal{K}\,(\phi, \mathbf{F}_{ab}, g_{ab}) = (\phi + k,\ \exp k\alpha\, \mathbf{F}_{ab}, g_{ab}) \tag{2.12}$$

where k is a real constant. It is easy to see that this transformation leaves the action invariant and hence also maps solutions to solutions. This symmetry is often exploited to restrict oneself to the sector of the theory in which $\phi_\infty = 0$. Again, properties of solutions with non-zero ϕ_∞ can be obtained by applying the appropriate map $\mathcal{K}$.

From now on, for notational simplicity, we set $Q_{\rm std}[S] = Q[S]$ and $Q_{\rm cons} = \tilde{Q}$

B. Static black holes

Static, spherically symmetric solutions to dilaton gravity are special cases of a general class of black holes first discovered by Gibbons [11] and Gibbons and Maeda [12]. (These solutions were independently discovered by Garfinkle, Horowitz and Strominger [13].) In the case $\alpha = 1$, they are the unique static black hole solutions of dilaton gravity [14]. For general α, we will refer to them as the Gibbons-Maeda (GM) solutions.

By making an appeal to the duality rotation $\mathcal{D}$ and the constant shift transformation $\mathcal{K}$, one restricts oneself to the case in which the magnetic charge P and the dilaton charge ϕ_∞ vanish. Then, the general solution is conveniently parameterized by two numbers, $r_\pm$ (with $\infty > r_+ \geq r_- > 0$), related to the ADM mass M and conserved electric charge $\tilde{Q}$ via:

$$GM = \frac{r_+}{2} + \left[\frac{1-\alpha^2}{1+\alpha^2}\right]\frac{r_-}{2} \tag{2.13}$$

$$G\tilde{Q}^2 = \left[\frac{r_-\,r_+}{1+\alpha^2}\right] \tag{2.14}$$

The solution is given by:

$$\mathrm{d}s^2 = -\lambda^2 \mathrm{d}t^2 + \lambda^{-2}\mathrm{d}r^2 + R^2 \mathrm{d}\Omega \tag{2.15}$$

$$\mathbf{F} = \frac{\tilde{Q}}{r^2}\,\mathrm{d}t \wedge \mathrm{d}r, \quad \text{and} \tag{2.16}$$

$$e^{2\alpha\phi} = \left[1 - \frac{r_-}{r}\right]^{2\alpha^2/(1+\alpha^2)}\ . \tag{2.17}$$

Here, r is a radial coordinate, related to the geometrical radius R of 2-spheres spanned by the orbits of the rotational Killing fields via

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[2]Note however that $\mathcal{D}$ is a *discrete* transformation and maps solutions with pure electric charge to those with pure magnetic charge. The analog of the continuous transformations $(\mathbf{F}, g) \mapsto (\cos\theta \mathbf{F} + \sin\theta\, {}^\star\mathbf{F},\ g)$ parameterized by an angle θ of the Einstein-Maxwell theory does not seem to exist in dilaton gravity. Furthermore, solutions with both electric and magnetic charge are known [10] only for $\alpha = 1$ and $\alpha = \sqrt{3}$, and they were *not* constructed using a symmetry of the theory .

$$R = r \left[1 - \frac{r_-}{r}\right]^{\alpha^2/(1+\alpha^2)} \tag{2.18}$$

and λ is a function of r, given by

$$\lambda^2 = \left[1 - \frac{r_+}{r}\right] \left[1 - \frac{r_-}{r}\right]^{(1-\alpha^2)/(1+\alpha^2)} . \tag{2.19}$$

When $\alpha = 0$, the dilaton field ϕ vanishes and the solution reduces to the Reissner-Nordström solution of the Einstein-Maxwell theory. The case $\alpha = 1$ is of interest to string theory. For all α, $r = r_+$ is an event horizon and for all *non-zero* values of α, there is a curvature singularity at $r = r_-$. For $\alpha = 0$, $r = r_-$ is a non-singular inner horizon. Finally, the extremal black holes correspond to $r_+ = r_-$ (or $Q^2 = G(1+\alpha^2)\, M^2$). The horizon of these extremal black holes is singular except when $\alpha = 0$.

In this 2-parameter family of black holes, one can directly compute the acceleration of the Killing field t^a (with $t^a\partial_a = \partial/\partial t$) on the horizon to obtain an expression for the surface gravity κ. We have:

$$\kappa = \frac{1}{2r_+} \left[1 - \frac{r_-}{r_+}\right]^{(1-\alpha^2)/(1+\alpha^2)} . \tag{2.20}$$

From (2.20) it is obvious that the zeroth law holds. Furthermore, $\kappa > 0$ for all non-extremal black holes. In the extremal case, since the event horizon is singular unless $\alpha = 0$, surface gravity is defined unambiguously only if $\alpha = 0$ (and is zero in this case). However, if $\alpha \leq 1$, one can define κ by a limiting procedure and show that $\kappa = 0$ for $\alpha \leq 1$ and $\kappa = \frac{1}{4M}$ if $\alpha = 1$.

Next, let us consider the first law. It is straightforward to check that, for any variation δ within this family of solutions, we have

$$\delta M \;=\; \frac{1}{8\pi G} \kappa\, \delta a_{\rm hor} + \Phi \delta \tilde{Q}\,, \tag{2.21}$$

where $a_{\rm hor}$ is the area of the horizon and $\Phi :\hat{=} \mathbf{A}_a t^a \equiv \tilde{Q}/r_+$ is the value of the natural scalar potential at the horizon.

To conclude, let us return to the symmetry transformations. Since the ADM mass M and the surface gravity κ are determined entirely by the metric and the metric does not change under $\mathcal{D}$ or $\mathcal{K}$, neither do M and κ. On the other hand, the electric and magnetic charges do change. The effect of $\mathcal{D}$ is the same as in Maxwell's theory. Let us therefore focus on $\mathcal{K}$. Let us begin with a Gibbons-Maeda solution $(\phi^o, \mathbf{F}^o_{ab}, g^o_{ab})$ and set $\mathcal{K}\,(\phi^o, \mathbf{F}^o_{ab}, g^o_{ab}) = (\phi, \mathbf{F}_{ab}, g_{ab})$. The resulting solution does not belong to the Gibbons-Maeda family since $\phi \mapsto k \neq 0$ at infinity. The parameters of the new solution are $(r_\pm = r^o_\pm,\, \phi_\infty = k)$. Thus, the (trivially) generalized Gibbons-Maeda (gGM) solutions are characterized by *three* parameters, $r_\pm, k$ with $0 < r_- \leq r_+ < \infty$ and $-\infty < k < \infty$, rather than just two. In terms of these, the physical quantities associated with the black-hole are given by:

$$M = M^o,\;\; a_{\rm hor} = a^o_{\rm hor},\;\; \kappa = \kappa^o,\;\; \Phi = \Phi^o\, e^{\alpha k} \;\text{ and }\; \tilde{Q} = \tilde{Q}^o\, e^{-\alpha k}. \tag{2.22}$$

Consequently, for this family, the first law becomes:

$$\delta M = \frac{1}{8\pi G}\,\kappa\,\delta a_{\rm hor} + \Phi\delta\tilde{Q} + \alpha\Phi\tilde{Q}\,\delta k$$
$$= \frac{1}{8\pi G}\,\kappa\delta a_{\rm hor} + \hat{\Phi}\delta\hat{Q} \qquad (2.23)$$

where $\hat{Q} = \tilde{Q}e^{\alpha k}$ (i.e. $\hat{Q}^2 = \tilde{Q}\,Q[S_\infty]$) and $\hat{\Phi} = \Phi e^{-\alpha k}$. We will return to these generalized Gibbons-Maeda solutions in Sections IV and V.

III. BOUNDARY CONDITIONS AND THEIR CONSEQUENCES

We are now ready to consider situations which are not necessarily static and introduce the notion of isolated horizons Δ for dilaton gravity. The basic boundary conditions defining Δ are the same as those introduced in [6,8]. However, we will use this opportunity to present them *without* recourse to spinors. While this formulation is less convenient for detailed calculations such as the ones performed in [8], it makes the meaning of the underlying assumptions more transparent.

Let us begin by introducing some notation. Fix any null surface $\mathcal{N}$, topologically $S^2 \times R$, and consider foliations of $\mathcal{N}$ by families of 2-spheres transversal to its null normal. Given a foliation, we parameterize its leaves by $v = \text{const}$ such that v increases to the future and set $n = -\mathrm{d}v$. Under a reparametrization $v \mapsto F(v)$, we have $n \mapsto F'(v)n$ with $F'(v) > 0$. Thus, every foliation comes equipped with an equivalence class $[n]$ of normals n related by rescalings which are constant on each leaf.[3] Now, given any one n, we can uniquely select a vector field ℓ which is normal to $\mathcal{N}$ and satisfies $\ell^a n_a = -1$. (Thus, ℓ is future-pointing. If we change the parameterization, ℓ^a transforms via: $\ell^a \mapsto (F'(v))^{-1}\ell^a$. Thus, given a foliation, we acquire an equivalence class $[\ell^a, n_a]$ of pairs, (ℓ^a, n_a) , of vector fields and 1-forms on $\mathcal{N}$ subject to the relation $(\ell^a, n_a) \sim (G^{-1}\ell, Gn)$, where G is any positive function on $\mathcal{N}$ which is constant on each leaf of the foliation. Given a pair (ℓ, n) in the equivalence class, we introduce a complex vector field m on $\mathcal{N}$, tangential to each leaf in the foliation, such that $m \cdot \overline{m} = 1$. (By construction, $m \cdot \ell = m \cdot n = 0$ on $\mathcal{N}$.) The vector field m is unique up to a phase factor. With this structure at hand, we now introduce the main Definition.

Definition: The internal boundary Δ of a space-time $(\mathbf{M}, g_{ab})$ will be said to represent *a non-rotating isolated horizon* provided the following conditions hold[4]:

- (i) *Manifold conditions:* Δ is a null surface, topologically $S^2 \times R$.
- (ii) *Dynamical conditions:* All field equations hold at Δ.

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[3]These 1-form fields n_a are defined intrinsically on $\mathcal{N}$. We can extend each n_a to the full space-time uniquely by demanding that the extended 1-form be null. However, in this paper, we will not need this extension.

[4]Throughout this paper, the symbol $\hat{=}$ will denote equality at points of Δ. For fields defined throughout space-time, an under-arrow will denote pull-back to Δ. The part of the Newman-Penrose framework [15] used in this paper is summarized in the Appendices A and B of [8].

- (iii) *Main conditions:* Δ admits a foliation such that the Newman-Penrose coefficients associated with the corresponding direction fields $[\ell, n]$ on Δ are subject to the following conditions:
  (iii.a) $\rho \,\widehat{=}\, -\overline{m}^a m^b \nabla_a \ell_b$, the expansion of $[\ell]$, vanishes on Δ.
  (iii.b) $\lambda \,\widehat{=}\, \overline{m}^a \overline{m}^b \nabla_a n_b$ and $\pi \,\widehat{=}\, \ell^a \overline{m}^b \nabla_a n_b$ vanish on Δ and the expansion $\mu := m^a \overline{m}^b \nabla_a n_b$ of n is negative [5] and constant on each leaf of the foliation.

- (iv) *Conditions on matter:* The Maxwell field $\mathbf{F}$ is such that

$$\phi_1 \,\widehat{=}\, \frac{1}{2}\, m^a \overline{m}^b (\mathbf{F} - i\, {}^\star\mathbf{F})_{ab} \tag{3.1}$$

  is constant on each leaf of the foliation introduced in condition (iii).

The first two conditions are quite tame: (i) simply asks that Δ be null and have appropriate topology while (ii) is completely analogous to the dynamical condition imposed at infinity. As the terminology suggests, (iii.a) and (iii.b) are the most important conditions. Note first that, if a pair (ℓ, n) in the equivalence class $[\ell, n]$ associated with the foliation satisfies these conditions, so does any other pair, $((G(v))^{-1}\ell,\, G(V)n)$. Thus, the conditions are well-defined. They are motivated by the following considerations. Condition (iii.a) captures the idea that the horizon is isolated without having to refer to a Killing field. In particular, it implies that the area of each 2-sphere leaf in the foliation be the same. We will denote this area by a_Δ and define the *horizon radius* R_Δ via $a_\Delta = 4\pi R_\Delta^2$. (In previous papers [6–8], the horizon radius was denoted by r_Δ. We have changed the notation to avoid possible confusion with the (non-geometric) radial coordinate r generally used in the discussion of dilaton black holes.)

Condition (iii.b) has three sets of implications. First, one can show that if, as required, one can find a foliation of Δ satisfying (iii.b), that foliation is *unique*. (In the gGM family, as one might expect, this condition selects the foliation to which the three rotational Killing fields are tangential.) Second, it implies that the imaginary part of (the Newman-Penrose Weyl component) Ψ_2, which captures angular momentum, vanishes and thus restricts us to *non-rotating* horizons. Third, the requirement that the expansion μ of n^a be negative implies that Δ is a *future* horizon rather than past [16]. Finally, consider the spherical symmetry requirement on the Maxwell field component ϕ_1. While this condition is a strong restriction, it can be motivated as follows. Conditions (i) – (iii) imply that $\phi_0 \,\widehat{=}\, -\ell^a m^b \mathbf{F}_{ab}$, the 'radiative part of the Maxwell field traversing Δ' vanishes (confirming the interpretation that Δ is isolated). If the 'radial derivative' of ϕ_0 also vanishes at Δ —i.e., heuristically, if there is no flux of electro-magnetic radiation across Δ also to the next order— condition (iv) is automatically satisfied. (For further motivation and remarks on these conditions, see [6,8].) Note that there is no explicit restriction on the dilaton field at Δ.

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[5] For simplicity, in this paper we focus on black-hole-type horizons rather than cosmological ones. To incorporate interesting cosmological horizons, one has to weaken this condition and allow the possibility that μ is everywhere positive on Δ. See [8].

Since these conditions are *local* to Δ, as indicated in the Introduction, the notion of an isolated horizon is quasi-local; in particular, one does not need an entire space-time history to locate an isolated horizon. Furthermore, the boundary conditions allow for presence of radiation in the exterior region. Therefore, space-times admitting isolated horizons need not admit any Killing field [17]. Indeed, the space $\mathcal{I}$ of solutions to field equations admitting isolated horizons is infinite dimensional. (Strategies for construction of such solutions are discussed in [8]).

In spite of this generality, boundary conditions place surprisingly strong restrictions on the structure of various fields *at* Δ. Let us begin with conditions on the dilaton and Maxwell fields. The stress-energy tensor T_{ab} of $(\phi, \mathbf{F})$ satisfies the dominant energy condition. Hence, on Δ, $-T_{ab}\ell^b$ is a future directed, causal vector field. Now, using the Raychaudhuri equation and field equations *at* Δ (condition (ii) of the Definition), we conclude $T_{ab}\ell^a\ell^b \,\hat{=}\, 0$. By expanding out this expression (see Eq (2.5)) we obtain

$$\dot{\phi} \equiv \mathcal{L}_\ell\, \phi \,\hat{=}\, 0 \quad \text{and} \quad \mathbf{F} \,\hat{=}\, \phi_1[\ell \wedge n - m \wedge \overline{m}] + \phi_2[m \wedge \ell] + \mathrm{CC} \tag{3.2}$$

for *some* complex functions ϕ_1 and ϕ_2 (the Newman-Penrose components of $\mathbf{F}$) on Δ, where CC stands for 'the complex conjugate term'. These equations say that that there is no flux of dilatonic or electro-magnetic radiation across Δ. Next, since $-T_{ab}\ell^a$ is a future pointing causal vector and $T_{ab}\ell^a\ell^b \,\hat{=}\, 0$, it follows that $T_{ab}\ell^a m^b \,\hat{=}\, 0$. Finally, the main conditions (iii) in the Definition imply that $R_{ab}m^a m^b \,\hat{=}\, 0$ [6,8] and, since the field equations hold at Δ, we conclude $T_{ab}m^a m^b \,\hat{=}\, 0$. This implies that the field ϕ is constant on Δ. We will denote this constant by ϕ_Δ and refer to it as *the dilatonic charge of the isolated horizon.* Finally, condition (iv) in the Definition implies

$$\phi_1 \,\hat{=}\, e^{2\alpha\phi_\Delta}\, \frac{2\pi}{a_\Delta}\, \tilde{Q}_\Delta\,, \tag{3.3}$$

where ϕ_Δ is the value of the dilaton field on Δ and $\tilde{Q}_\Delta \equiv \tilde{Q}$ is the conserved electric charge. Thus the boundary conditions severely restrict the form of matter fields at Δ. The dilaton field ϕ is constant on Δ, the component $\phi_0 = -\ell^a m^b \mathbf{F}_{ab}$ of the Maxwell field vanishes and the component ϕ_1 is completely determined by the dilaton and (conserved) electric charge. However, the component ϕ_2 of the electro-magnetic field is unconstrained.

Restrictions imposed on space-time curvature at Δ are essentially the same as in Ref [8].[6] Results relevant to this paper can be summarized as follows. In the Newman-Penrose notation, for the Ricci tensor components, we have:

$$\begin{aligned} &\Phi_{00} = \frac{1}{2}\, R_{ab}\ell^a\ell^b \,\hat{=}\, 0, \quad \Phi_{01} = \frac{1}{2}\, R_{ab}\ell^a m^b \,\hat{=}\, 0, \Phi_{02} = \frac{1}{2}\, R_{ab}m^a m^b \,\hat{=}\, 0, \\ &\Phi_{11} = \frac{1}{4} R_{ab}(\ell^a n^b + m^a \overline{m}^b) \,\hat{=}\, -8\pi^2 G\, e^{2\alpha\phi_\Delta} \left(\frac{\tilde{Q}_\Delta}{a_\Delta}\right)^2, \quad R \,\hat{=}\, 0, \end{aligned} \tag{3.4}$$

[6]This is because these restrictions were obtained assuming rather general conditions on the matter stress-energy which are satisfied in dilaton gravity. The derivation of some of these results involve long calculations and a topological result on the Chern-class of the $SO(2)$ connection associated with the dyad $(m, \overline{m})$. See [8].

where R is the scalar curvature. The Weyl tensor components satisfy

$$\Psi_0 = C_{abcd}\ell^a m^b \ell^c m^d \,\hat{=}\, 0, \qquad \Psi_1 = C_{abcd}\ell^a m^b \ell^c n^d \,\hat{=}\, 0$$
$$\Psi_2 = C_{abcd}\ell^a m^b \overline{m}^c n^d \,\hat{=}\, \Phi_{11} - \frac{2\pi}{a_\Delta} \tag{3.5}$$

Furthermore,

$$\Psi_3 \,\hat{=}\, \Phi_{21}, \quad \text{that is} \quad C_{abcd}\ell^a n^b \overline{m}^c n^d \,\hat{=}\, \frac{1}{2} R_{ab}\overline{m}^a n^b. \tag{3.6}$$

This structure is the same as in the gGM solutions discussed at the end of Section II. However, on a general isolated horizon, other curvature components may be 'dynamical', i.e., vary along the integral curves of ℓ.

In view of this structure of various fields on Δ, it is natural to use the triplet $R_\Delta, \tilde{Q}_\Delta, \phi_\Delta$ —the horizon radius, the conserved charge and the value of the dilaton field on the horizon— to parameterize general isolated horizons. In the gGM solutions, these parameters are subject to the restriction $R_\Delta^2 \geq (1+\alpha^2)G\tilde{Q}^2 \exp(2\alpha\phi_\Delta)$. In the more general context of isolated horizons, we will restrict the range of parameters by the same condition. Unlike the ADM mass M, the total electric charge Q_∞ and the dilatonic charge ϕ_∞, these parameters are *local* to Δ. In the special case of gGM solutions, this triplet is uniquely determined by the parameters $r_\pm, k \equiv \phi_\infty$ used in Section III and, reciprocally, determine $r_\pm, k$ uniquely.

We will conclude this section with a remark. There exist in the literature other families of static, spherically symmetric solutions with Maxwell and dilaton fields with Killing horizons which, however, fail to be asymptotically flat (or anti-de Sitter) [18]. In the general case, the field equations they obey contain a potential term $V(\phi)$ for the dilaton field and reduce to those given in Section II A only in the case when $V(\phi) = 0$. However, for all potentials, our boundary conditions are satisfied at the Killing horizons of these solutions; thus those horizons are, in particular, isolated horizons. Furthermore, for *general* isolated horizons in any of these theories, even without Killing fields, all conclusions of this section continue to hold (even though, for certain potentials $V(\phi)$, the stress-energy tensor does not satisfy the dominant energy condition). Isolated horizons offer a natural home for such black holes since, unlike event horizons, they do not refer to the structure at infinity.

IV. SURFACE GRAVITY AND THE ZEROTH LAW

In each gGM solution there is a unique time-translational Killing field t^a which is unit at infinity. As usual, surface gravity $\kappa_{\rm GM}$ is defined in terms of its acceleration at the horizon: $t^a\nabla_a t^b \,\hat{=}\, \kappa_{\rm GM}\, t^b$. In terms of the gGM parameters $(r_\pm, k)$, $\kappa_{\rm GM}$ is given by (2.20). From the perspective of the isolated horizon framework, κ is the acceleration of the properly normalized null normal ℓ^a to Δ [7,8]. In the gGM solutions, Δ happens to be a Killing horizon and we can select a unique vector field ℓ^a from the the equivalence class $[\ell^a]$ simply by setting $\ell^a \,\hat{=}\, t^a$. Then $\kappa_{\rm GM}$ is the acceleration of this specific ℓ^a. In the case of general isolated horizons, the challenge is to find a prescription to single out a preferred ℓ^a, without reference to any Killing field. The strategy we adopt is identical to that used for isolated

horizons without a dilaton coupling in [7,8]. However, unlike in those references, we will now proceed in two steps to bring out a general conceptual issue.

In the first step, we will normalize ℓ *only* up to a constant, leaving a rescaling freedom $\ell \mapsto \ell' = c\ell$, where c is a constant on Δ but may depend on the parameters $R_\Delta, \tilde{Q}_\Delta, \phi_\Delta$ of the isolated horizon. For each such ℓ, we can define the surface gravity κ_ℓ *relative to that* ℓ via $\ell^a \nabla_a \ell^b \,\hat{=}\, \kappa_\ell \ell^b$. Rescaling of ℓ now induces to a 'gauge transformation' in κ: $\kappa_\ell \mapsto \kappa_{\ell'} = c\kappa_\ell$. (Recall that in the general Newman-Penrose framework, κ is a connection component and therefore undergoes the standard gauge transformations under a change of the null tetrad. By fixing ℓ up to a constant rescaling, we have reduced the general gauge freedom to that of a constant rescaling.) Since the zeroth law only says that the surface gravity is constant on Δ, if it holds for one ℓ, it holds for every $\ell' = c\ell$. Thus, for the zeroth law, it is in fact *not* essential to get rid of the rescaling freedom.

Recall that the isolated horizon is naturally equipped with equivalence classes $[\ell, n]$ of vector and co-vector fields, subject to the relation: $(\ell, n) \sim (G^{-1}\ell, Gn)$ for any positive function $G \equiv G(v)$ on Δ. Our first task is to reduce the freedom in the choice of $G(v)$ to that of a constant. We use the same strategy as in [7,8]. (For motivation, see [8].) Recall that μ, the expansion of n is strictly negative and constant on each leaf of the preferred foliation; $\mu \equiv \mu(v) < 0$. It is easy to verify that

$$n^a \mapsto G(v) n^a \text{ implies } \quad \mu \mapsto G(v)\mu(v). \tag{4.1}$$

Hence, we can *always* use the $G(v)$ freedom to set $\mu \,\hat{=}\, \text{const}$. This condition restricts the family of (ℓ, n) pairs and reduces the equivalence relation to $(\ell, n) \sim (c\ell, c^{-1}n)$ where c is any constant on Δ. We will denote the restricted equivalence class by $[\ell, n]_R$. In the second step, we will *fix* the numerical value of μ in terms of the parameters of the isolated horizon and eliminate the rescaling freedom altogether, thereby selecting a canonical pair (ℓ, n) on each isolated horizon.

With the equivalence class $[\ell, n]_R$ at our disposal, as discussed above, we can define a surface gravity κ_ℓ via $\ell^a \nabla_a \ell^b \,\hat{=}\, \kappa_\ell \ell^b$. Constancy of κ_ℓ on Δ follows from the same arguments that were used in [8]. For completeness, let us briefly recall the structure of that proof. First, using conditions on derivatives of l, n introduced in the Definition, one can express the self-dual part of the Riemann curvature in terms of $\kappa_\ell, \mathrm{d}\kappa_\ell, \mu$ (and another field which is not relevant to this discussion). Comparing this expression to the standard Newman-Penrose expansion of the self-dual curvature tensor in terms of curvature scalars [15], and using the fact that certain curvature scalars vanish on Δ (see (3.4) and (3.5)), one can conclude

$$\mathrm{d}\kappa_\ell \wedge n \,\hat{=}\, 0, \quad \text{and} \quad \kappa_\ell \,\hat{=}\, \frac{\Psi_2}{\mu}. \tag{4.2}$$

The first equation implies that κ_ℓ is spherically symmetric. Hence, it only remains to show that $\mathcal{L}_\ell \kappa_\ell \,\hat{=}\, 0$. Since μ is now a constant on Δ, it suffices to show that $\mathcal{L}_\ell \Psi_2 = 0$. Now, the (second) Bianchi identity implies that

$$\mathcal{L}_\ell \left(\Psi_2 - \Phi_{11}\right) \,\hat{=}\, 0 \tag{4.3}$$

Finally, using (3.2) and the fact that the dilaton field ϕ is constant on Δ, we conclude: $\Phi_{11} = -8\pi^2 G \left(\exp(2\alpha\phi_\Delta)\right) (\tilde{Q}_\Delta / a_\Delta)^2$. Thus, Φ_{11} is constants on Δ. Combining these results, we conclude $\mathcal{L}_\ell \kappa_\ell \,\hat{=}\, 0$, whence κ_ℓ is constant on Δ. This establishes the zeroth law.

With an eye toward the first law, let us now carry out the second step in fixing the normalization of ℓ. So far, we have only required that μ be a (negative) constant but not fixed its value. Under the rescaling $\mu \mapsto c^{-1}\mu$ we have: $\ell \mapsto c\ell$, and $\kappa_\ell \mapsto c\kappa_\ell$. Hence, the remaining rescaling freedom in ℓ and κ can be exhausted simply by fixing the value of μ in terms of the isolated horizon parameters. The obvious strategy is to fix μ to the value $\mu_{\rm gGM}$ it takes on the gGM solutions. However, there is a practical difficulty: Although $\mu_{\rm gGM}$ is a well-defined function of the isolated horizon parameters $R_\Delta, \tilde{Q}_\Delta$ and ϕ_Δ, to our knowledge, there is no closed expression for it in terms of these parameters.

Therefore, for computational convenience, let us first make a change of coordinates on the three dimensional parameter space $\mathcal{P}$ defined by the triplet $(R_\Delta, \tilde{Q}_\Delta, \phi_\Delta)$. Introduce r_Δ and k_Δ via

$$\begin{aligned} R_\Delta &= r_\Delta \left[1 - (1+\alpha^2)\, \frac{G\tilde{Q}^2_\Delta e^{2\alpha k_\Delta}}{r^2_\Delta}\right]^{\alpha^2/1+\alpha^2} \\ e^{\alpha\phi_\Delta} &= \frac{R_\Delta}{r_\Delta}\, e^{\alpha k_\Delta}. \end{aligned} \tag{4.4}$$

It is easy to check that $(r_\Delta, \tilde{Q}_\Delta, k_\Delta)$ is a good set of coordinates on $\mathcal{P}$ with ranges $r^2_\Delta > G\tilde{Q}^2_\Delta$, $-\infty < \tilde{Q}_\Delta < \infty$ and $-\infty < k_\Delta < \infty$. With these definitions, in the gGM solutions we have:

$$r_\Delta = r_+, \quad \tilde{Q}^2_\Delta = \left[\frac{r_- r_+}{1+\alpha^2}\right] e^{-2\alpha k}, \quad \text{and} \quad k_\Delta = k\,. \tag{4.5}$$

However, from a conceptual point of view, on general isolated horizons $(R_\Delta, \tilde{Q}_\Delta, \phi_\Delta)$ are the fundamental parameters and $(r_\Delta, \tilde{Q}_\Delta, k_\Delta)$ are simply convenient functions of them defined via (4.4). In particular, on a general isolated horizon, k_Δ is unrelated to the value ϕ_∞ of the dilaton field at infinity.

We are now ready to express $\mu_{\rm gGM}$ of (2.20) in a closed form:

$$\mu_{\rm gGM} \mathrel{\widehat{=}} -\frac{1}{r_\Delta}\left[1 - \frac{G\tilde{Q}^2 e^{2\alpha k_\Delta}}{r^2_\Delta}\right]\left[1 - (1+\alpha^2)\, \frac{G\tilde{Q}^2 e^{2\alpha k_\Delta}}{r^2_\Delta}\right]^{-1} \tag{4.6}$$

Therefore, for *general isolated horizons* we will gauge-fix ℓ by demanding that μ be given by the right side of (4.6). With this final 'gauge-fixing', ℓ is uniquely determined on each isolated horizon. We will denote the corresponding κ_ℓ simply by κ and call it *the surface gravity* of the isolated horizon Δ. Using (4.2), (3.4) and (3.5) and the expression (4.6) of μ, we can express κ in terms of the isolated horizon parameters as

$$\kappa = \frac{2\pi}{a_\Delta}\, r_\Delta \left[1 - (1+\alpha^2)\, \frac{G\tilde{Q}^2_\Delta\, e^{2\alpha k_\Delta}}{r^2_\Delta}\right], \tag{4.7}$$

where, as before, $a_\Delta = 4\pi R^2_\Delta$ is the area of the horizon. By construction, in the gGM solutions, κ reduces to (2.20).

Finally, as discussed in [8], to arrive at the standard form of the first law, we need to gauge fix the electro-magnetic potential $\Phi \mathrel{\widehat{=}} \mathbf{A}\cdot\ell$. Again, the strategy is similar to the one

we used to gauge fix the normalization of ℓ: We first express $\Phi_{\rm gGM}$ in the gGM solutions in terms of the isolated horizon parameters and then ask that Φ be the same function of the parameters in the general case. Again, to obtain an expression of $\Phi_{\rm gGM}$ in a closed form, we are led to use $(r_\Delta, \tilde{Q}_\Delta, k_\Delta)$. It is easy to verify that

$$\Phi_{\rm gGM} = \frac{\tilde{Q}_\Delta}{r_\Delta}\, e^{2\alpha k_\Delta}\,. \tag{4.8}$$

Hence, on general isolated horizons, we work in a gauge in which Φ is given by the right side of (4.8). As in the Einstein-Maxwell case [8], this gauge fixing makes the matter action $S_{\rm Dil}$ functionally differentiable even in presence of the internal boundary Δ.

We conclude this section with a remark. A general isolated horizon carries three independent parameters, $(R_\Delta, \tilde{Q}_\Delta, \phi_\Delta)$, arising from the metric, Maxwell and dilaton fields respectively. Therefore, to unambiguously implement the above procedure to gauge-fix the normalization of ℓ, we needed access to a three parameter family of 'reference' static solutions, one for each point in the parameter set. This is why it was essential to consider the generalization of the Gibbons-Maeda solutions to include non-zero values of ϕ_∞. Had we restricted ourselves to the original Gibbons-Maeda family, the mapping between the 3-dimensional parameter space $\mathcal{P}$ and the space of static solutions would have been ambiguous. For the generalized Gibbons-Maeda family, the mapping is 1-1 and hence unambiguous.

V. MASS AND THE FIRST LAW

As explained in the Introduction, since space-times under consideration are not necessarily stationary, it is no longer meaningful to identify the ADM mass M with the mass M_Δ of the isolated horizon. For the formulation of the first law, therefore, we must first introduce an appropriate definition of M_Δ. The Hamiltonian framework provides a natural strategy. In the Einstein-Maxwell case the total Hamiltonian consists of a bulk term and *two* surface terms, one at infinity and the other at the isolated horizon. As usual, the bulk term is a linear combination of constraints and the surface term at infinity yields the ADM energy. In a rest-frame adapted to the horizon it is then natural to identify the surface term at Δ as the horizon mass, M_Δ. Indeed, there are several considerations that support this identification [8]. We will use the same strategy in the dilatonic case.

For the gravitational part of the action and Hamiltonian, the discussion of [8] only assumed that the stress-energy tensor satisfies two conditions at Δ: i) $-T_{ab}\ell^a$ is a future pointing causal vector field on Δ; and, ii) $T_{ab}\ell^a n^b$ is spherically symmetric on Δ. Both these conditions are met in the present case. Therefore, we can take over the results of [8] directly. For the matter part of the action and Hamiltonian, the overall situation is again analogous, although there are the obvious differences in the detailed expressions because of dilaton couplings. As in the Einstein-Maxwell case, matter terms contribute to the surface terms in the Hamiltonian only because one has to perform one integration by parts to obtain the Gauss constraint in the bulk term.

The net result is the following. Consider a foliation of the given space-time region $\mathbf{M}$ by a 1-parameter family of (partial) Cauchy surfaces M, each of which extends from the isolated horizon Δ to spatial infinity i^o (see Figure). We will assume that M intersects Δ

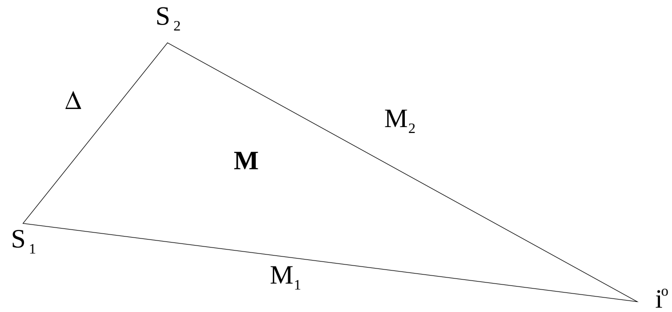

FIG. 1. Region **M** of space-time considered in the variational principle is bounded by two partial Cauchy surfaces M_1 and M_2. They intersect the isolated horizon Δ in preferred 2-spheres S_1 and S_2 and extend to spatial infinity i^o.

in a 2-sphere belonging to our preferred foliation and that the initial data induced on M are asymptotically flat. Denote by S_Δ and S_∞ the 2-sphere boundary of M at the horizon and infinity, respectively. Choose a time-like vector field t^a in $\mathbf{M}$ which tends to the unit time-translation orthogonal to the foliation at spatial infinity and to the vector field ℓ^a on Δ, normalized as in Section IV. Then, the Hamiltonian H_t generating evolution along t^a is given by:

$$H_t = \int_M \text{constraints} \ + \lim_{R_o\to\infty} \oint_{S_{R_o}} \left(\frac{R_o}{4\pi G}\,\Psi_2\right)\ {}^2\epsilon \\ + \oint_{S_\Delta} \left(\frac{\mu^{-1}}{4\pi G}\Psi_2\right){}^2\epsilon + (\mathbf{A}\cdot l)\tilde{Q}_\Delta \tag{5.1}$$

where S_{R_o} are large 2-spheres of radius R_o. (The calculation and the final result are completely analogous to those in the Einstein-Maxwell case [8]. The only differences are: i) Since we have set the cosmological constant equal to zero, the scalar curvature of the 4-metric vanishes on Δ, and ii) the surface term that results from integration by parts of the Maxwell Gauss constraint now contains $\tilde{Q}_\Delta$ in place of Q_Δ.) Note that the surface terms depend only on the 'Coulombic' parts of the gravitational and Maxwell fields. The form of their integrands is rather similar since asymptotically the Newman-Penrose spin-coefficient μ goes as R_o^{-1}. However, while the surface term at infinity depends *only* of the Weyl curvature, the term at the horizon depends also on the Maxwell Potential.

It is easy to check that the surface term at infinity is, as usual, the time component $P_a^{\rm ADM}t^a$ of the ADM 4-momentum $P_a^{\rm ADM}$, which in the present -,+,+,+ signature is negative of the ADM energy, $P_a^{\rm ADM}t^a = -E^{\rm ADM}$. It is natural to identify the surface term at S_Δ as the energy of the isolated horizon. (There is no minus sign because S_Δ is the *inner* boundary of M). Since $t^a \,\hat{=}\, \ell^a$ and since ℓ^a represents the 'rest frame' of the isolated horizon, this energy can in turn be identified with the horizon mass M_Δ. Thus, we have:

$$M_\Delta = \oint_{S_\Delta} \left(\frac{\mu^{-1}}{4\pi G}\Psi_2\right){}^2\epsilon + (\mathbf{A}\cdot l)\tilde{Q}_\Delta\,. \tag{5.2}$$

Using the expression (4.2) of surface gravity in terms of the Weyl tensor and the expression (4.8) of $\mathbf{A}\cdot\ell$ on Δ, we can cast M_Δ in a more familiar form:

$$M_\Delta = \frac{1}{4\pi G}\,\kappa a_\Delta \,+\, \Phi \tilde{Q}_\Delta \tag{5.3}$$

Thus, as in the gGM solutions, we obtain the Smarr formula. However, the meaning of various symbols in the equation is somewhat different. Since an isolated horizon need not be a Killing horizon, in general M_Δ does not equal the ADM mass, nor is κ or Φ computed using a Killing field. If the space-time happens to be static with t^a as the Killing field, general arguments from symplectic geometry can be used to show that the numerical value of the Hamiltonian H_t vanishes at this solution [8]. Since the constraints are satisfied in any solution, the bulk term in (5.1) vanishes as well. Hence, in this case, $M_\Delta = E^{\rm ADM}$. In general, however, the two differ by the 'radiative energy' in the space-time. In the Einstein-Maxwell case, one can argue [8] that, if Δ stretches all the way to i^+, under suitable regularity conditions M_Δ equals the future limit of the Bondi-energy in the frame in which the black hole is at rest. There appears to be no obstruction to extending that argument to the dilatonic case. Finally, as emphasized in [8], the matter contribution to the mass formula (5.2) is subtle: while it does not include the energy in radiation outside the horizon, it does include the energy in the 'Coulombic part' of the field associated with the black hole hair. (Recall that the future limit of the Bondi energy has this property.) However, since this issue was discussed in detail in [8], and since dilaton couplings do not add substantive complications, we will not repeat that discussion it here.

We now have expressions of the mass M_Δ, surface gravity κ, area a_Δ and the electric potential Φ of any isolated horizon in terms of the convenient functions $r_\Delta, \tilde{Q}_\Delta, k_\Delta$ of its fundamental parameters $R_\Delta, \tilde{Q}_\Delta, \phi_\Delta$:

$$\begin{aligned} M_\Delta &= \frac{r_\Delta}{2G}\left(1 + (1-\alpha^2)\frac{G\tilde{Q}_\Delta^2\, e^{2\alpha k_\Delta}}{r_\Delta^2}\right) \quad ; \quad a_\Delta = 4\pi r_\Delta^2 \left(1 - (1+\alpha^2)\,\frac{G\tilde{Q}_\Delta^2\, e^{2\alpha k_\Delta}}{r_\Delta^2}\right)^{2\alpha^2/(1+\alpha^2)} \\ \kappa &= \frac{2\pi r_\Delta}{a_\Delta}\left(1 - (1+\alpha^2)\frac{G\tilde{Q}_\Delta^2\, e^{2\alpha k_\Delta}}{r_\Delta^2}\right) \quad ; \quad \Phi = \frac{\tilde{Q}_\Delta}{r_\Delta}\, e^{2\alpha k_\Delta}\, . \end{aligned} \tag{5.4}$$

Since all quantities are now expressed in a closed form in terms of the same parameters of isolated horizons, it is easy to compute their variations. A simple calculation yields[7]

$$\begin{aligned} \delta M_\Delta &= \frac{1}{8\pi G}\,\kappa\delta a_\Delta + \Phi\delta\tilde{Q}_\Delta + \alpha\Phi\tilde{Q}_\Delta\,\delta k_\Delta \\ &= \frac{1}{8\pi G}\,\kappa\delta a_\Delta + \hat{\Phi}\delta\hat{Q}_\Delta \end{aligned} \tag{5.5}$$

where $\hat{\Phi} = \Phi e^{-\alpha k_\Delta}$ (i.e., $\hat{\Phi}^2 = \tilde{Q}Q[S_\Delta]/R_\Delta^2$) and $\hat{Q} = \tilde{Q}e^{\alpha k_\Delta}$. Thus, as one might have expected from the Einstein-Maxwell analysis of [8], the first law (2.23) for general isolated horizons has the same form as (2.23), the first law in gGM solutions. The only difference in the detailed expression is the replacement of the ADM mass M by the horizon mass M_Δ.

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[7]A similar expression of the first law with contributions to the 'work term' from scalar fields was first obtained in [19]. That analysis was restricted only to *static solutions* but carried out on a more general context of theories with *several* scalar fields.

However, physically, this difference is important [8]. Consider, for example, a space-time which admits an isolated horizon Δ_1 for a finite time interval which ceases to be isolated for a small duration because of the influx of radiation into the horizon and settles down again to an isolated horizon Δ_2. Then, using two 'triangular' regions $\mathbf{M_1}$ and $\mathbf{M_2}$ in the space-time with internal boundaries Δ_1 and Δ_2 respectively (see Fig. 1), one can compute the masses M_{Δ_1} and M_{Δ_2} of the two isolated horizons. While the ADM mass M of the space-time remains unchanged during this physical process, the isolated horizon mass does change; $M_{\Delta_1} \neq M_{\Delta_2}$. By using the Raychaudhuri equation, one can show that the the difference is precisely the mass associated with the radiation which fell in. Thus, in this process the mass contained in the radiation outside the horizon decreases, the horizon mass increases but the sum, which equals the ADM mass M, remains unchanged. If there is flux of charged matter across the horizon, the situation is more subtle but the final result is the same. As discussed in detail in [8], in this case, there does not appear to exist a treatment of the physical process version of the first law without recourse to the isolated horizon framework.

Finally, note that the expression (5.3) of M_Δ can be trivially rewritten in terms of the hatted variables:

$$M_\Delta = \frac{1}{4\pi G}\,\kappa a_\Delta \,+\, \hat{\Phi}\hat{Q}_\Delta \tag{5.6}$$

Thus, for dilatonic isolated horizons, the smarr formula and the first law have the same form as in the Einstein-Maxwell case, the only difference being that the electric potential and charge are now replaced by $\hat{\Phi}$ and $\hat{Q}$.

We will conclude this discussion with three remarks:

a) The space $\mathcal{IH}$ of space-times admitting isolated horizons is infinite dimensional. Fortunately, however, the boundary conditions are sufficiently strong to enable us to express physical parameters such as the mass M_Δ and the surface gravity κ in terms of only three parameters; they are lifts to $\mathcal{IH}$ of functions on $\mathcal{P}$. Explicit calculations presented in this section made a heavy use of the new coordinates $(r_\Delta, \tilde{Q}_\Delta, k_\Delta)$ on $\mathcal{P}$. However, from a conceptual standpoint, the introduction of these coordinates is not essential. For, M_Δ, κ, a_Δ and Φ are all well-defined functions on $\mathcal{P}$ and the first law is a statement of relations between the gradients of three of these functions. It makes no reference at all to the choice of coordinates on $\mathcal{P}$. Indeed, in principle, we could have used only the fundamental parameters $(R_\Delta, \tilde{Q}_\Delta, \phi_\Delta)$ throughout; passage to $(r_\Delta, \tilde{Q}_\Delta, k_\Delta)$ served to simplify the algebra.

b) The new parameterization entered our explicit calculations because we wished to reduce the rescaling freedom in ℓ in such a way that in the static (gGM) solutions ℓ coincides with the restriction to Δ of that static Killing field which is unit at infinity. To implement this condition, we had to ask that, on general isolated horizons, μ be the same function of parameters as $\mu_{\rm gGM}$ is in static solutions and $\mu_{\rm gGM}$ could not be expressed in a closed form in terms of $(R_\Delta, \tilde{Q}_\Delta, \phi_\Delta)$. While this choice of scale for ℓ is most convenient for making contact with the standard static framework, we could have made other choices. In general, if we rescale ℓ via $\ell \mapsto \ell' = c\ell$, where c is a constant on Δ (but may be a function of the parameters), we have $\mu \mapsto \mu' = c^{-1}\mu$, $M_\Delta \mapsto M'_\Delta = cM_\Delta$, $\kappa \mapsto \kappa' = c\kappa$ and $\Phi \mapsto \Phi' = c\Phi$ while the area a_Δ and the charge $\tilde{Q}_\Delta$ remain unchanged (see Eq (5.2) and the discussion in Section IV).

Let us choose $c = R_\Delta / r_\Delta$. Then we have

$$M'_\Delta = \frac{R_\Delta}{2G}\left(1 + (1-\alpha^2)\frac{GQ'^2_\Delta}{R^2_\Delta}\right) \quad ; \quad a'_\Delta = 4\pi R^2_\Delta$$

$$\kappa' = \frac{2\pi R_\Delta}{a_\Delta}\left(1 - (1+\alpha^2)\frac{GQ'^2_\Delta}{R^2_\Delta}\right) \quad ; \quad \Phi' = \frac{Q[S_\Delta]}{R_\Delta}\,, \tag{5.7}$$

where, as before, $Q[S_\Delta] = \tilde{Q}_\Delta\, e^{2\alpha\phi_\Delta}$ and $Q'^2_\Delta = \tilde{Q}_\Delta Q[S_\Delta]$. Thus, in this rescaling gauge, all physical quantities are expressed entirely in terms of the fundamental parameters $(R_\Delta, \tilde{Q}_\Delta, \phi_\Delta)$. In this gauge the first law becomes:

$$\delta M'_\Delta = \frac{1}{8\pi G}\kappa'\,\delta a'_\Delta + (1-\alpha^2)\hat{\Phi}'\delta Q'_\Delta\,, \tag{5.8}$$

where $\hat{\Phi}' = Q'_\Delta/R_\Delta$. Note that Eqs (5.7) and (5.8) are completely equivalent to Eqs (5.4) and (5.5); the two sets express the same information but in different rescaling-gauges. One set is adapted to the fundamental parameters of isolated horizons but fails to reduce to the standard set of expressions used in the static context. The other set does reproduce the standard expressions in the static case but involves the use of auxiliary parameters $(r_\Delta, \tilde{Q}_\Delta, k_\Delta)$.

c) Our main results can be immediately generalized to incorporate a cosmological constant Λ. The only essential change in Section III is that the scalar curvature R of the 4-metric at the horizon is now given by $R = 4\Lambda$, which in turn introduces factors of Λ in the expressions of physical quantities in Sections IV and V (see [8]). Now, it is known that there are no static black hole solutions in presence of a positive cosmological constant [20]. Therefore, if isolated horizons exist with $\Lambda > 0$, we would not be able to fix the normalization of ℓ using the procedure used in Section IV. However, as the discussion of Sections IV and V and the discussion in the point b) above shows, one could still establish the zeroth and first laws. For $\Lambda < 0$, static solutions are known to exist [20] and we can repeat the procedure given above. Then, one can show

$$\kappa = \frac{2\pi r_\Delta}{a_\Delta}\left(1 - (1+\alpha^2)\frac{G\tilde{Q}^2}{r^2_\Delta}e^{2\alpha k_\Delta} - \frac{a_\Delta}{4\pi}\Lambda\right) \tag{5.9}$$

The electrostatic potential Φ is unaffected and the Smarr formula (5.3) and the first law continues to hold.

VI. DISCUSSION

In [8], the zeroth and first laws were established for mechanics of non-rotating isolated horizons in general relativity. While the restrictions on the matter content were rather mild, that analysis did not allow a dilatonic charge or dilaton couplings. In this note, we extended that analysis to dilaton gravity. Key ideas and strategies are the same as those introduced in [8]. However, the incorporation of general dilaton couplings —i.e., arbitrary values of the coupling parameter α— was not entirely straightforward because the specific gauge choices needed to define surface gravity and electro-magnetic potential on general isolated horizons are now rather complicated. However, once an appropriate viewpoint

towards parameterization of general isolated horizons is introduced, one realizes that the complications are rather tame; only the algebra becomes more involved. In the final picture, the proof of the zeroth law is completely analogous to that in the Einstein-Maxwell case and the first law has exactly the same form as in static gGM solutions.

There are three natural directions in which the results of this paper could be extended. First, as in the previous work [6,8], our boundary conditions imply that the intrinsic 2-metric on Δ is spherically symmetric (although the space-time metric need not admit *any* Killing field in a neighborhood of Δ.) It is of interest to extend the analysis to incorporate (non-rotating but) distorted isolated horizons. This would only require weakening of the boundary condition (iii.b) on derivatives of n^a and (iv) on spherical symmetry of ϕ_1. A second and more interesting extension would be inclusion of rotation. Again, only conditions (iii.B) and (iv) would have to be weakened. Very recently, both these extensions were carried out for the Einstein-Maxwell theory [21,22]. Furthermore, in contrast to the present paper, these analyses do not fix the scaling of ℓ^a using static solutions; a more general strategy is involved. Therefore, it is likely that those considerations will extend to dilaton couplings as well, even though explicit distorted or rotating dilatonic solutions are still not known. Finally, as pointed out at the end of Section III, isolated horizons provide a natural home to study black holes with non-standard asymptotic structure. For theories considered in [18], the present analysis already suffices to establish the zeroth law, i.e., the constancy of κ on the horizon. It would be interesting to analyze whether the methods developed in [21,22] can be used to establish the first law.

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